Exercise 4

Use residues to evaluate the improper integrals in Exercises 1 through 8.

$$\int_0^\infty \frac{x \sin 2x}{x^2+3}\,dx.$$

$$\textit{Ans.}\ \frac{\pi}{2}\exp(-2\sqrt{3}).$$

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{x \sin 2x}{x^2+3}\,dx = \int_{-\infty}^\infty \frac{x \sin 2x}{2(x^2+3)}\,dx$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{ze^{2iz}}{2(z^2+3)},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$2(z^2+3) = 0$$
$$z^2+3 = 0$$
$$z = \pm i\sqrt{3}$$

The singular point of interest to us is the one that lies within the closed contour, $z = i\sqrt{3}$.

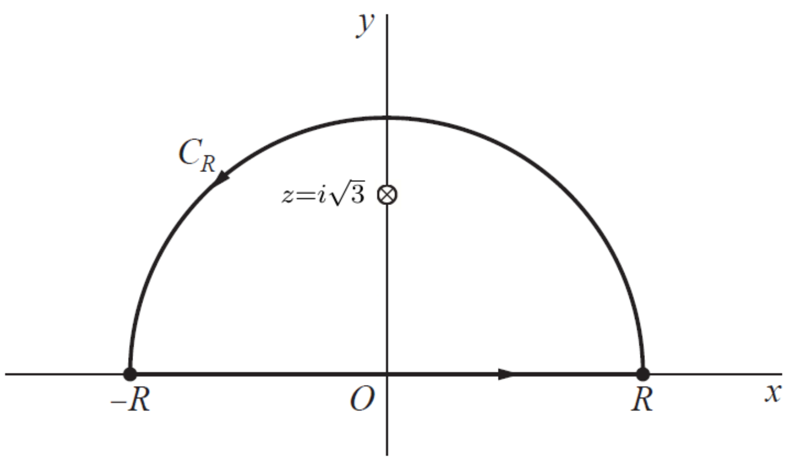

Figure 1: This is Fig. 93 with the singularity at $z = i\sqrt{3}$ marked.

According to Cauchy's residue theorem, the integral of $ze^{2iz}/[2(z^2+3)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{ze^{2iz}}{2(z^2+3)}\,dz = 2\pi i \operatorname*{Res}_{z=i\sqrt{3}} \frac{ze^{2iz}}{2(z^2+3)}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{ze^{2iz}}{2(z^2+3)}\,dz + \int_{C_R} \frac{ze^{2iz}}{2(z^2+3)}\,dz = 2\pi i \operatorname*{Res}_{z=i\sqrt{3}} \frac{ze^{2iz}}{2(z^2+3)}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, && r=-R \quad \to \quad r=R \\ C_R: &\quad z = Re^{i\theta}, && \theta = 0 \quad \to \quad \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{re^{2ir}}{2(r^2+3)}\,dr + \int_{C_R} \frac{ze^{2iz}}{2(z^2+3)}\,dz = 2\pi i \operatorname*{Res}_{z=i\sqrt{3}} \frac{ze^{2iz}}{2(z^2+3)}.$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{re^{2ir}}{2(r^2+3)}\,dr = 2\pi i \operatorname*{Res}_{z=i\sqrt{3}} \frac{ze^{2iz}}{2(z^2+3)}$$

The denominator can be written as $2(z^2+3) = 2(z+i\sqrt{3})(z-i\sqrt{3})$. From this we see that the multiplicity of the $z - i\sqrt{3}$ factor is 1. The residue at $z = i\sqrt{3}$ can then be calculated by

$$\operatorname*{Res}_{z=i\sqrt{3}} \frac{ze^{2iz}}{2(z^2+3)} = \phi(i\sqrt{3}),$$

where $\phi(z)$ is equal to $f(z)$ without the $z - i\sqrt{3}$ factor.

$$\phi(z) = \frac{ze^{2iz}}{2(z+i\sqrt{3})} \quad \Rightarrow \quad \phi(i\sqrt{3}) = \frac{i\sqrt{3}e^{2i^2\sqrt{3}}}{2(2i\sqrt{3})} = \frac{e^{-2\sqrt{3}}}{4}$$

So then

$$\operatorname*{Res}_{z=i\sqrt{3}} \frac{ze^{2iz}}{2(z^2+3)} = \frac{\exp(-2\sqrt{3})}{4}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{re^{2ir}}{2(r^2+3)}\,dr &= 2\pi i \left[\frac{\exp(-2\sqrt{3})}{4}\right] \\ \int_{-\infty}^{\infty} \frac{r\cos 2r + ir\sin 2r}{2(r^2+3)}\,dr &= \frac{i\pi}{2}\exp(-2\sqrt{3}) \\ \int_{-\infty}^{\infty} \frac{r\cos 2r}{2(r^2+3)}\,dr + i\int_{-\infty}^{\infty} \frac{r\sin 2r}{2(r^2+3)}\,dr &= \frac{i\pi}{2}\exp(-2\sqrt{3}). \end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{r\cos 2r}{2(r^2+3)}\,dr = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{r\sin 2r}{2(r^2+3)}\,dr = \frac{\pi}{2}\exp(-2\sqrt{3})$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{x\sin 2x}{x^2+3}\,dx = \frac{\pi}{2}\exp(-2\sqrt{3}).}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{ze^{2iz}}{2(z^2+3)}\,dz &= \int_0^\pi \frac{Re^{i\theta}e^{2iRe^{i\theta}}}{2[(Re^{i\theta})^2+3]}(Rie^{i\theta}\,d\theta)\\ &= \int_0^\pi \frac{e^{2iR(\cos\theta+i\sin\theta)}}{R^2e^{i2\theta}+3}\left(\frac{R^2ie^{i2\theta}}{2}\,d\theta\right)\\ &= \int_0^\pi \frac{e^{2iR\cos\theta}e^{-2R\sin\theta}}{R^2e^{i2\theta}+3}\left(\frac{R^2ie^{i2\theta}}{2}\,d\theta\right)\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{ze^{2iz}}{2(z^2+3)}\,dz\right| &= \left|\int_0^\pi \frac{e^{2iR\cos\theta}e^{-2R\sin\theta}}{R^2e^{i2\theta}+3}\left(\frac{R^2ie^{i2\theta}}{2}\,d\theta\right)\right|\\ &\le \int_0^\pi \left|\frac{e^{2iR\cos\theta}e^{-2R\sin\theta}}{R^2e^{i2\theta}+3}\left(\frac{R^2ie^{i2\theta}}{2}\right)\right|d\theta\\ &= \int_0^\pi \frac{\left|e^{2iR\cos\theta}\right|\left|e^{-2R\sin\theta}\right|}{|R^2e^{i2\theta}+3|}\left|\frac{R^2ie^{i2\theta}}{2}\right|d\theta\\ &= \int_0^\pi \frac{e^{-2R\sin\theta}}{|R^2e^{i2\theta}+3|}\frac{R^2}{2}\,d\theta\\ &\le \int_0^\pi \frac{e^{-2R\sin\theta}}{|R^2e^{i2\theta}|-|3|}\frac{R^2}{2}\,d\theta\\ &= \int_0^\pi \frac{e^{-2R\sin\theta}}{R^2-3}\frac{R^2}{2}\,d\theta\\ &= \int_0^\pi \frac{e^{-2R\sin\theta}}{1-\frac{3}{R^2}}\frac{d\theta}{2}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{2iz}}{2(z^2+3)}\,dz\right| \le \lim_{R\to\infty}\int_0^\pi \frac{e^{-2R\sin\theta}}{1-\frac{3}{R^2}}\frac{d\theta}{2}$$

Because the limits of integration do not depend on R, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{2iz}}{2(z^2+3)}\,dz\right| \le \int_0^\pi \lim_{R\to\infty}\frac{e^{-2R\sin\theta}}{1-\frac{3}{R^2}}\frac{d\theta}{2}$$

Since θ lies between 0 and π, the sine of θ is positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{2iz}}{2(z^2+3)}\,dz\right| \le 0$$

The magnitude of a number cannot be negative, and the only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{2iz}}{2(z^2+3)}\,dz\right| = 0 \quad \to \quad \lim_{R\to\infty}\int_{C_R} \frac{ze^{2iz}}{2(z^2+3)}\,dz = 0.$$